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LETTER TO THE EDITOR

Monte Carlo renormalisation group approach to multifractal structure of growth probability distribution in DLA

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Abstract. A Monte Carlo renormalisation group method is presented to study the fractal structure of a DLA cluster. This method converges very rapidly with an increase in the size of the cell. Applying this technique to the DLA grown on the square lattice, the scaling structure of the growth probability distribution in the surface layer is calculated. An infinite set of generalised dimensions $D(q)$ and the α-f spectra are found. We estimate that the maximum value of the generalised dimension and the surface fractal dimension are $D_\infty = 0.5 + 0.05$ and $D_0 = 1.5 + 0.05$, up to the scale factor $b = 30$, in excellent agreement with the conjecture by Ball.

The diffusion-limited aggregation (DLA) cluster is one of the most intensively studied random fractals (Witten and Sander 1981, Meakin 1983, Family and Landau 1984, Stanley and Ostrowsky 1985, Pietronero and Tosatti 1986, Stanley 1986, Herrmann 1986). Theoretical, computational and experimental results were systematically analysed using the scaling ideas. The renormalisation group (RG) method provides a general framework for concepts such as universality and scaling in critical phenomena (Wilson and Kogut 1974, Burkhardt and van Leeuwen 1982). Gould *et al* (1983) have derived the fractal dimension for DLA by making use of the position-space renormalisation group method. Kolb (1987) has made an attempt at a Monte Carlo renormalisation for the growth process. Since there was no Hamiltonian formulation of DLA, they adopted the position-space renormalisation group approach in which the change in connectivity of the cluster was determined upon repeated length rescaling.

It is well known that an aggregate cannot be fully characterised only by its fractal dimension. Meakin *et al* (1986), Halsey *et al* (1986) and Amitrano *et al* (1986) have recently shown that the surface of DLA requires an infinite hierarchy of fractal dimensions for its characterisation. They have presented the multifractal structure of the growth probability distribution from numerical experiments. Nagatani (1987a, b, c) has presented a real-space renormalisation group method for calculating the multifractality of the growth probability distribution where a small-cell renormalisation was applied with the scale factor $b = 2, 3$. Hayakawa *et al* (1987) have calculated the multifractality on the off-lattice by using the more accurate numerical technique. Ohta and Honjo (1988) have presented the α-f spectra from the solidification experiment.

In this letter, we present a Monte Carlo renormalisation method to derive the multifractality of the growth probability distribution. We restrict ourselves to the DLA problem on the square lattice. We extend the small-cell renormalisation method to larger-cell renormalisation because small-cell renormalisation gives somewhat inaccurate critical exponents.

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We describe the DLA process in a discrete electrostatic language. The process is characterised by the theorem of minimum dissipation energy. We consider the dissipation energy in place of the Hamiltonian. The dissipation energy is given by

$$H_d = \sum_i \sigma_i E_i^2$$

$$= E_0\left(\sum_{i_s} \sigma_{i_s} E_{i_s}\right) \tag{1}$$

where σ_i indicates the conductance of bond i, E_i is the electric field on the bond i and E_0 is the total voltage drop of the system. The first expression represents the summation of dissipation energy on each bond over total bonds within the system and the second expresses this in terms of the surface conductance σ_{i_s}, where the total current carrying through the surface layer of the aggregate is given by $J_t = \Sigma_{i_s}\, \sigma_{i_s} E_{i_s}$. By using a decimation method, one may write the dissipation energy in terms of the coarse-grained variables σ'_{i_s}, E'_{i_s} on the new lattice with the lattice spacing b (b is the scale factor):

$$H_d = E_0\left(\sum_{i_s} \sigma'_{i_s} E'_{i_s}\right). \tag{2}$$

The surface conductance σ is transformed to σ'. This surface conductance plays the role of the so-called coupling constant in the Hamiltonian of phase transitions. We can obtain the renormalisation group equation:

$$\sigma' = R(\sigma). \tag{3}$$

On the small-cell renormalisation, (3) has a single stable fixed point, consistent with the minimum principle of the dissipation energy. We consider the renormalisation procedure for deriving the RG equation (3). We define the bonds on the surface as the growth bonds which can be successively grown. The surface conductance is defined as the conductance of the growth bond. Cover all the space of the square lattice by cells of edge b (scale factor), each containing $2b^2$ bonds. After a renormalisation transformation these cells play the role of 'renormalised' bonds. On the renormalised lattice, the surface conductance is transformed from the bare value to the renormalised value. The renormalised surface conductance will be assumed to be given by the most probable value

$$\sigma' = \exp\left(\sum_\alpha C_\alpha \ln \sigma'_\alpha\right) \tag{4}$$

where C_α represents the probability of a particular configuration α appearing and σ'_α is the total conductance of the cell with the configuration α. After renormalisation, the growth probability $P_i(L)$ on any growth bond i is given by

$$P_i(L) = p_{\beta,i} P_\beta(L/b) \tag{5}$$

where L represents the size of the system, b is the scale factor and $p_{\beta,i}$ indicates the growth probability of the growth bond i within the cell β. The cell's growth probability $p_{\beta,i}$ is represented by a function of the surface conductance, depending on the configuration of the cell. After many repeated renormalisations, the growth probability assigned to each growth bond is represented by a random multiplicative process of the cell's

growth probabilities evaluated at the fixed point. In the limit of L sufficiently large, an infinite hierarchy of generalised dimensions $D(q)$ is given by

$$D(q) = -(q-1)^{-1} \ln \left\langle \sum_i p_{\alpha,i}^{*q} \right\rangle (\ln b)^{-1} \tag{6}$$

where $\langle\ \rangle$ indicates the configurational average and $p_{\alpha,i}^{*}$ represents the cell's growth probability evaluated at the fixed point.

By making use of a Monte Carlo technique, we evaluate the configurational average in (4) and (6). A cell of the type used here is shown in figure 1(a). We explain a procedure to make a Monte Carlo realisation for the growth bond. We apply a unit voltage between the top and bottom of the cell shown in figure 1(a). The dielectric breakdown proceeds from the bottom to the top. So the electrostatic problem is solved under the boundary condition by using the relaxation method. A growth probability proportional to the current is then assigned to the perimeter bond (growth bond). The interface proceeds to the top according to the growth probability, where the breakdown occurs one by one. The breakdown process continues until the interface of the aggregate connects with the top of the cell (see figure 1). The break bond and the growth bond are indicated by the bold and wavy lines. The pattern of the cell, obtained at each step of the breakdown process, corresponds to a configuration of the cell which is renormalised as the growth bond. This breakdown process is repeated N times. The total conductance of the cell and the growth probability distribution within the cell are calculated at each step of the breakdown. The configurational averages in (4) and (6) are approximated by averaging over $N \times M$ Monte Carlo realisations:

$$\begin{aligned} \langle \ln \sigma \rangle &\simeq (1/N)(1/M) \sum_{n=1}^{N} \sum_{m=1}^{M} \ln \sigma_{nm} \\ \left\langle \sum p_i^q \right\rangle &\simeq (1/N)(1/M) \sum_{n=1}^{N} \sum_{m=1}^{M} \left(\sum p_i^q \right)_{nm} \end{aligned} \tag{7}$$

where M indicates the number of steps until the aggregate reaches the top of the cell. The number of bonds within the aggregate, when the breakdown stops, is given by

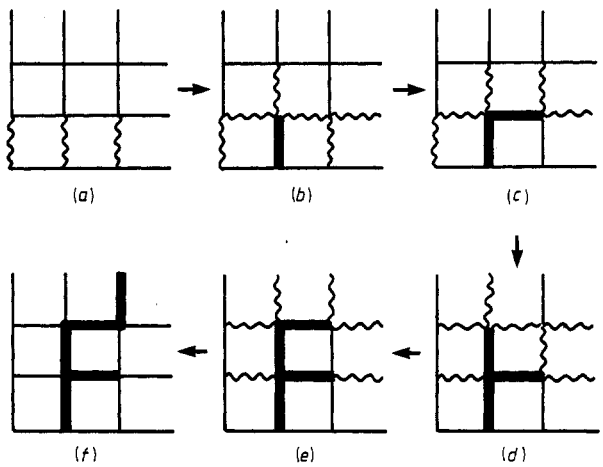

Figure 1. Illustration of the dielectric breakdown process within the cell ($b = 3$) on the square lattice. Break, growth and unbroken bonds are respectively indicated by bold, wavy and light lines. The breakdown process proceeds one by one ($a \to b \to c \to d \to e \to f$).

$M+1$. The surface conductance σ^* at the fixed point is found by the iteration method. Firstly, we assume an appropriate value of surface conductance. We find the most probable value of the total conductance of the cell. Secondly, by replaying the initial value with the obtained value, we continue to obtain iteratively the value of the surface conductance. The surface conductance approaches the fixed value. This convergence of the surface conductance is granted by the stable fixed point consistent with the minimum theorem of the dissipation energy. By using the surface conductance obtained here as the fixed point, we can obtain the growth probability of the cell via (7). Table 1 shows the generalised dimensions $D(q)$ for $b=5$, 10, 15, 20, 30.

Table 1. Values of D_q for DLA obtained from the Monte Carlo renormalisation group method.

b	5	10	15	20	30
N	100	100	30	20	10
D_{-5}	5.522	6.381	6.274	6.158	6.655
D_{-4}	5.187	6.028	5.939	5.862	6.381
D_{-3}	4.695	5.511	5.446	5.424	5.975
D_{-2}	3.922	4.682	4.658	4.715	5.310
D_{-1}	2.664	3.212	3.302	3.416	4.012
D_0	1.458	1.465	1.477	1.486	1.505
D_1	1.033	1.030	1.039	1.043	1.047
D_2	0.936	0.939	0.942	0.945	0.957
D_3	0.843	0.855	0.864	0.868	0.888
D_4	0.778	0.793	0.806	0.807	0.832
D_5	0.731	0.744	0.760	0.755	0.782
D_6	0.693	0.702	0.720	0.709	0.736
D_7	0.663	0.667	0.685	0.671	0.695
D_8	0.637	0.635	0.654	0.639	0.662
D_9	0.614	0.608	0.626	0.612	0.634
D_{10}	0.595	0.583	0.602	0.590	0.611
D_{20}	0.478	0.451	0.472	0.484	0.507

Our method converges very rapidly with an increase in the size of the cell. We compare our result with other work (Amitrano *et al* 1986, Hayakawa *et al* 1987, Ohta and Honjo 1988). Our result and that of Amitrano *et al* are obtained for the square lattice. The results of Hayakawa *et al* and Ohta and Honjo were obtained for off-lattice. The surface fractal dimension D_0 agrees with Amitrano *et al.* In our result, the surface fractal dimension is also equivalent to $D_{\infty+1}$. However, our result obtained from lattice DLA does not agree with that obtained from off-lattice DLA. For the fractal dimension $d_f = D_\infty + 1$, our result agrees with the conjecture ($d_f = \frac{3}{2}$) by Ball (1986). The information dimension D_1 agrees with the other work.

The α-f spectra for $b=30$ are shown in figure 2 by the full curve. Ohta and Honjo's result obtained from experimental data is indicated by the broken curve. The height of the curve is a little lower than the off-lattice result. There is the difference between our result and Ohta and Honjo's data. This is due to the difference between the lattice and the off-lattice.

In summary, we present the Monte Carlo renormalisation method to derive the multifractal structure of the growth probability distribution for DLA on the square

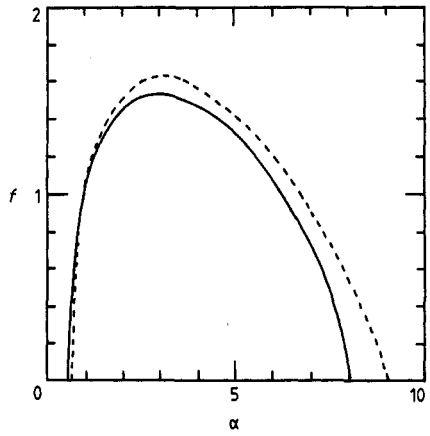

Figure 2. The α-f spectra for DLA. The full curve indicates our result on the square lattice. The broken curve represents Ohta and Honjo's experimental data.

lattice. This method gives quick convergence with an increase in the cell size. With relatively small cells, we obtain values in excellent agreement with the large-size Monte Carlo simulation result.

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